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Lower Semicontinuous Regularization for Vector-Valued Mappings

Dedicated to Alex Rubinov in honor of his 65th birthday.

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Abstract. The paper is devoted to studying the lower semicontinuity of vector-valued mappings. The main object under consideration is the lower limit. We first introduce a new definition of an adequate concept of lower and upper level sets and establish some of their topological and geometrical properties. A characterization of semicontinuity for vector-valued mappings is thereafter presented. Then, we define a concept of vector lower limit, proving its lower semicontinuity, and furnishing in this way a concept of lower semicontinuous regularization for mappings taking their values in a complete lattice. The results obtained in the present work subsume the standard ones when the target space is finite dimensional. In particular, we recapture the scalar case with a new flexible proof. In addition, extensions of usual operations of lower and upper limits for vector-valued mappings are explored. The main result is finally applied to obtain a continuous D.C. decomposition of continuous D.C. mappings.

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1. Introduction

The concept of lower semicontinuity introduced for scalar functions by R. Baire has been recognized as a fundamental tool in different areas of mathematical analysis. It has been used in different contexts and in particular by D. Hilbert and L. Tonelli in the calculus of variations. The rapid development of optimization theory, in particular Pareto optimization, has made evident the necessity of extending this concept to vector-valued mappings. This has motivated a quite number of mathematicians to investigate this topic. The first attention in this direction goes back to Théra [24], Gierz et al. [11], Penot and Théra [19], Gerrits [10], Holwerda [12], Luc [14], Borwein and Théra [4], Combari et al. [6]. We refer also to the recent contributions by Akian [1] and Akian and Singer [2].

A basic fact in real analysis is that every real-valued function f admits a lower semicontinuous regularization, l.s.c regularization for short, defined by means of the lower limit of f :

$$\bar{f}(x) := \liminf_{y \to x} f(y). \tag{1.1}$$

A very natural and challenging question is, therefore, to determine a concept of l.s.c regularization for vector-valued mappings. It seems that, since the contribution of Théra [24], in which he provided some types of l.s.c regularizations for mappings with values in order complete Daniell spaces and lattices Daniell spaces, a little bit of attention has been focused on the topic. Therefore, there is still a need to make some advances in this direction in the general setting of mappings with values in partially ordered spaces not necessarily Daniell.

The main scope of this paper is then to define an appropriate l.s.c regularization for mappings with values in a complete Banach lattice. Thus, after defining a suitable lower limit, our efforts will mostly be devoted in proving its semicontinuity.

Inspired by the ideas of Penot and Théra [19] and motivated by Combari et al. [6], we introduce a concept of lower and upper "level" sets. We first study these sets, show that they own nice properties, both topological and geometrical, and establish the link between them and semicontinuity. Then, we succeed in defining the concept of lower limit for a vector-valued mapping f at a point x in its domain. We will use the notation $v - \liminf_{y \to x} f(y)$ rather than the standard one $\liminf_{y \to x} f(y)$ in order to make clear that we are in the framework of vector-valued mappings.

We now outline the plan of the remaining contents of the work, which we organize in nine sections. Section 2 includes the notations and most necessary definitions used later. Section 3 introduces adequate lower and upper level sets which are illustrated by some examples. The fourth section deals with characterizations of semicontinuity for vector-valued mappings. Section 5 presents a study of some of the topological and geometrical properties of the lower level set. In Section 6, we reach our goal by proving that the vector lower limit we consider defines a l.s.c regularization for a given vector-valued map. We begin with Hilbert-valued maps and then consider Banach lattice-valued ones. In Section 7 we check that our contribution can be viewed as an extension of standard results. In particular, we recapture the scalar case with a more flexible proof. In Section 8, we extend the usual operations of estimation of lower and upper limits of the sum of two vector-valued mappings. Section 9 aims to apply the main result to obtain a continuous decomposition for D.C.-mappings in the vector case, extending in this way some previous results in the literature.

2. Notations and Definitions

Throughout this paper, E and F are real-vector topological spaces. For a subset S in E or F, $\operatorname{Int} S$ and $\operatorname{cl} S$ denote the interior and the closure of S, respectively. Let $C \subset F$ be a closed and convex cone, which is supposed to be pointed, that is $C \cap -C = \{0\}$, and with nonempty interior. The cone C defines a partial order on F denoted by $\leqslant_c$ and defined by

$$y_1 \leqslant_c y_2 \Leftrightarrow y_2 \in y_1 + C. \tag{2.1}$$

We also write $y_1 \nleqslant_c y_2$ whenever $y_2 - y_1 \notin C$. The positive polar cone C_+^* of F is defined by

$$C_+^* = \left\{ y^* \in F^* : \left\langle y^*, y \right\rangle \geqslant 0 \text{ for all } y \in C \right\}, \tag{2.2}$$

where F^* is the continuous dual of F and $\langle ., . \rangle$ the corresponding duality pairing. The set C_0 used in the sequel is defined by

$$C_0 := \{ w \in C : \xi(w) > 0, \ \ \forall \xi \in C_+^* \backslash \{0\} \}.$$

Remark 2.1. Notice that we have $\operatorname{Int} C \subset C_0$, we refer for instance to [17].

$F^\bullet$ will stand for $F \cup \{+\infty\}$, where $+\infty$ denotes the greatest element of F with respect to the order $\leqslant_c$. We will write $x <_c y$, for $x, y \in E$, if $y - x \in \operatorname{Int} C$. The order between subsets in F is defined as follows:

DEFINITION 2.2. Let A and B be two subsets of F. We write $A \leqslant_c B$, if for each $x \in A$ and each $y \in B$, we have $y - x \in C$.

It is equally worth to recall that a subset A of F is said to be *directed upwards* if for every a, b in A there exists $c \in A$ such that $a \leqslant_c c$ and $b \leqslant_c c$. By analogy, *directed downwards* subsets can be defined.

For a given subset $A \subset F$, there may, or may not, be $a \in F$ with the following property :

$$\text{for every } c \in F, \ \ a \leqslant_c c \text{ if and only if } \ b \leqslant_c c \text{ for every } b \in A,$$

that is, $a \leqslant_c c$ if and only if, c is an upper bound for A. Obviously, if a exists it is unique; it is called the *least upper bound* of A and denoted by $\sup_{\leqslant_c} A$ or simply $\sup A$ if there is no risk of confusion on the order. In a similar way, $\inf A$, whenever it exists, is called the *greatest lower bound* of A and it is the element of F such that

$$\text{for every } c \in F,\ c \leqslant_c \inf A \text{ if and only if } c \leqslant_c b \text{ for every } b \in A.$$

We shall recall that F is called a *lattice* whenever $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all elements a, b in F. It follows that finite subsets in a lattice have a least upper bound and a greatest lower bound. Finally, notice that the class of lattices such that non-empty subsets bounded above and directed upwards have a least upper bound contains the subclass of complete lattices (also called *Dedekind complete lattices*).[1] We point out here that no confusion should be done between complete lattices and totally ordered spaces.

The domain of a function, $f: E \to F^{\bullet}$, is denoted by $\mathrm{Dom}\, f$ and is defined by

$$\mathrm{Dom}\, f = \{x \in E \mid f(x) <_c +\infty\},$$

and its epigraph by

$$\mathrm{epi}\, f = \{(x, y) \in E \times F \mid y \in f(x) + C\}. \tag{2.3}$$

Recall now the following definitions:

DEFINITION 2.3. f is said to be C-convex, if for every $\alpha \in [0, 1]$ and $x_1, x_2 \in E$ one has

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha) x_2) + C. \tag{2.4}$$

DEFINITION 2.4. A mapping $f: E \to F$ is said to be C-D.C., if there exist two C-convex mappings g and h such that:

$$f(x) = g(x) - h(x) \quad \forall x \in E.$$

The pair (g, h) of C-convex maps will be called a C-D.C. decomposition of f.

We recall now the definitions of lower semicontinuity and sequential lower semicontinuity of a vector-valued mapping introduced respectively in [19] and [6].

[1]Some authors use the terminology of conditionally complete lattices as in [2].

DEFINITION 2.5. [19] A mapping $f:E\to F^\bullet$ is said to be *lower semicontinuous* (l.s.c) at $\bar{x}\in E$, if for any neighborhood V of zero and for any $b\in F$ satisfying $b\leqslant_c f(\bar{x})$, there exists a neighborhood U of $\bar{x}$ in E such that

$$f(U)\subset b+V+C\ \cup\{+\infty\}.$$

Remark 2.6. Following [19], if $f(\bar{x})\in F$ then Definition 2.5 amounts to saying that for all neighborhood V of zero (in F), there exists a neighborhood U of $\bar{x}$ such that

$$f(U)\subset f(\bar{x})+V+C\cup\{+\infty\}. \tag{2.5}$$

DEFINITION 2.7. [6] A mapping $f:E\to F^\bullet$ is said to be *sequentially lower semicontinuous* (s-l.s.c) at $\bar{x}\in E$, if for any $b\in F$ satisfying $b\leqslant_c f(\bar{x})$ and for any sequence (x_n) in E which converges to $\bar{x}$, there exists a sequence (b_n) (in F) converging to b and satisfying $b_n\leqslant_c f(x_n)$, for every $n\in\mathbb{N}$.

Remark 2.8. For $\bar{x}\in \operatorname{Dom} f$, Definition 2.7 can be expressed simply as follows: For each sequence (x_n) converging to $\bar{x}$, there exists a sequence (b_n) converging to $f(\bar{x})$ such that $b_n\leqslant_c f(x_n)$ for all $n\in\mathbb{N}$.

Note that it has been proved in [6] that Definitions 2.5 and 2.7 coincide whenever E and F are metrizable.

DEFINITION 2.9. F is said to be *normal* if F has a basis of order-convex neighborhood of zero of the form $V=(V+C)\cap(V-C)$.

Remark 2.10. It is worth mentioning as well that:

- The sequential upper semicontinuity of f(s-u.s.c for brevity) is defined by saying that $-f$ is s-l.s.c.
- If (F,C) is *normal*, one may check that f is sequentially continuous at $\bar{x}\in E$ with $f(\bar{x})\in F$, if and only if f is s-l.s.c and s-u.s.c at $\bar{x}$.
- Whenever E is metrizable and $F=\mathbb{R}$, the s-l.s.c continuity coincides with the classical lower semicontinuity. In this case, a function is s-l.s.c at every point of E if and only if its epigraph is closed in $E\times F$.
- Note that every l.s.c vector-valued mapping has a closed epigraph (see [4]), but the converse is not true as the following counterexample furnished in [19] shows:

The mapping $h:\mathbb{R}\to\mathbb{R}^2$ defined by

$$h(x)=\begin{cases}(0,0) & \text{if } x=0\\ \left(\frac{1}{|x|},-1\right) & \text{otherwise,}\end{cases}$$

is not not l.s.c at $(0,0)$ while its epigraph (with respect to $C=\mathbb{R}^2_+$) is closed.

We end up these preliminaries by recalling that the lower part of the Painlevé–Kuratowski set-convergence of a sequence (A_n) of subsets of F, is given by

$$\liminf_n A_n=\{y\in F: y=\lim_n y_n,\ \text{there exists}\, n_0:\ \text{for all}\ n\geqslant n_0,\ y_n\in A_n\}.$$

The sequence $(A_n)_n$ will be said lower convergent to $A\subset F$ in the sense of Painlevé–Kuratowski if and only if

$$A\subset\liminf_n A_n.$$

3. Adequate Local Lower and Upper Level Sets

In the present section, on the way to our objective, we introduce adequate notions of local lower and upper "level" sets for vector-valued mappings defined from E into $F^\bullet$.

DEFINITION 3.1. Let f be an extended-vector-valued mapping, $\overline{x}\in\operatorname{Dom} f$ and $y\in F$. Denoting by $\vartheta(\bar{x})$ (resp. $\vartheta(y)$) the family of neighborhoods of $\overline{x}$ (resp. y), we introduce the following "level" sets:

$$A^f_{\bar{x}}:=\{y\in F\mid \forall\, V\in\vartheta(y),\ \exists U\in\vartheta(\bar{x}),\ f(U)\subset V+C\cup\{+\infty\}\};\qquad (3.1)$$

$$B^f_{\bar{x}}:=\{y\in F\mid \forall\, V\in\vartheta(y),\ \exists U\in\vartheta(\bar{x}),\ f(U)\subset V-C\cup\{+\infty\}\};\qquad (3.2)$$

$$s-A^f_{\bar{x}}:=\{y\in F\mid \forall(x_n)_n\to\bar{x},\ \exists(b_n)_n\to y,\ b_n\leqslant_c f(x_n)\ \forall n\in\mathbb{N}\};\qquad (3.3)$$

$$s-B^f_{\bar{x}}:=\{y\in F\mid \forall(x_n)_n\to\bar{x},\ \exists(b_n)_n\to y,\ b_n\geqslant_c f(x_n)\ \forall n\in\mathbb{N}\}.\qquad (3.4)$$

Remark 3.2. In order to illustrate these definitions we present the following examples:

EXAMPLE 1. Let f be the real-valued function defined by

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x+1 & \text{otherwise.} \end{cases}$$

Note that f is not lower semicontinuous and observe that

$$A_x^f = \begin{cases}]-\infty, x] & \text{if } x \leqslant 0 \\ ]-\infty, x+1] & \text{otherwise.} \end{cases}$$

Remark 3.3. Let us point out that for any extended real-valued function $f, \mathrm{cl}\, A_{\bar{x}}^f =]-\infty, \liminf_{x\to\bar{x}} f(x)]$. In Section 7 we present the proof in details for finite dimensional-valued functions.

EXAMPLE 2. Let H be a separable Hilbert space and let $(e_n)_{n\in}$ be a basis of H. We suppose that the order is defined by the closed convex cone H_+ given by

$$H_+ = \{x \in H \mid \langle e_p, x\rangle \geqslant 0,\ \forall p \in \mathbb{N}\}.$$

Denoting $\langle e_i, f\rangle$ by f_i, we consider the function $f : \mathbb{R} \longrightarrow H$ defined by

$$\begin{cases} f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases} & \\ f_i(x) = 0 & \text{if } i \neq 1. \end{cases}$$

We observe that f is lower semicontinuous everywhere except at 0, and we check that[2]

$$A_x^f = \prod_{p=1}^{\infty} \left\langle e_p, A_x^f \right\rangle.$$

Indeed,

$$\left\langle e_p, A_x^f \right\rangle = A_x^{\langle e_p, f\rangle}.$$

Then, for $p > 1$, we have

$$A_x^{\langle e_p, f\rangle} = A_x^{f_p} =]-\infty, 0].$$

[2]as it is done later in Lemma 6.2.

For $p=1$, we see that

$$A_x^{\langle e_1, f\rangle} = A_x^{f_1} =]-\infty, 0].$$

Hence,

$$A_x^f = \prod_{p=1}^{\infty} \langle e_p, A_x^f\rangle = \prod_{p=1}^{\infty}]-\infty, 0] = -H_+.$$

EXAMPLE 3. Consider the mapping $f=(f_1, f_2)\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$\begin{cases} f_1(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{otherwise.} \end{cases} \\ f_2(x) = x. \end{cases}$$

A straightforward calculation shows that $\mathrm{cl}\, A_x^f =]-\infty, 0] \times]-\infty, x]$ if $x \leqslant 0$ and $\mathrm{cl}\, A_x^f =]-\infty, 1] \times]-\infty, x]$ otherwise.

Remark that $\mathrm{cl}\, A_x^f$ is convex, directed upwards and in addition we have

$$A_x^f = A_x^f - \mathbb{R}_+^2.$$

In Section 5, we will prove that these properties of this lower "level" set hold in general.

In the next section, we characterize the semicontinuity of vector-valued mappings in terms of the above level sets.

4. Characterization of Semicontinuity for Vector-Mappings

We begin with the following proposition which gives the link between the level sets and the s-l.s.c, l.s.c, s-u.s.c and u.s.c.

PROPOSITION 4.1. *Let f be an extended vector map and $\bar{x} \in \mathrm{Dom}\, f$. Then,*

(1) *f is s-l.s.c at $\bar{x}$ if and only if, $f(\bar{x}) \in s - A_{\bar{x}}^f$;*
(2) *f is l.s.c en $\bar{x}$ if and only if, $f(\bar{x}) \in A_{\bar{x}}^f$;*
(3) *f is s-u.s.c at $\bar{x}$ if and only if, $f(\bar{x}) \in s - B_{\bar{x}}^f$;*
(4) *f is s.c.s at $\bar{x}$ if and only if, $f(\bar{x}) \in B_{\bar{x}}^f$.*

Proof. The proof follows from the definitions. □

Now, we can characterize the level sets of semicontinuous extended-vector-valued mappings. First, we establish the equivalence between s-$A_{\bar{x}}^f$ and $A_{\bar{x}}^f$ once E and F are metrizable.

PROPOSITION 4.2. *Assume that* E *and* F *are metrizable. Let* $f: E \to F^\bullet, \bar{x} \in \mathrm{Dom}\, f$. *Then, we have*

$$s - A^f_{\bar{x}} = A^f_{\bar{x}} \text{ and } s - B^f_{\bar{x}} = B^f_{\bar{x}}.$$

Proof. For the inclusion $s - A^f_{\bar{x}} \subseteq A^f_{\bar{x}}$, suppose to the contrary that there is some $y \in s - A^f_{\bar{x}} \backslash A^f_{\bar{x}}$. There exists then a neighborhood V of y and a sequence $(x_n)_n$ converging to $\bar{x}$ such that $f(x_n) \notin V + C$. Hence, for every sequence $(b_n)_n$ converging to y, one has $b_n \not\leqslant_c f(x_n)$ for n sufficiently large. This contradicts the fact that $y \in s - A^f_{\bar{x}}$.

For the converse inclusion, let $y \in A^f_{\bar{x}}$ and let $(x_n)_n$ be a sequence converging to $\bar{x}$. For $k = 1$, one can find $n_1 > 1$ such that

$$f(x_n) \in y + B(0, 1) + C \cup \{+\infty\}, \quad n \geqslant n_1. \tag{4.1}$$

By induction, for $k > 1$ there exists $n_k \geqslant \cdots \geqslant n_1$ such that

$$f(x_n) \in y + B(0, \frac{1}{k}) + C \cup \{+\infty\}, \quad n \geqslant n_k. \tag{4.2}$$

Now, for n with $0 < n \leqslant n_1$ set $b_n = f(x_n)$, for n with $n_k < n \leqslant n_{k+1}$, $k = 1, 2, \ldots$ choose $b_n \in y + B(0, \frac{1}{k})$ such that $b_n \leqslant_c f(x_n)$ according to (4.1) and (4.2). Then, $(b_n)_n$ converges to y and hence $y \in s - A^f_{\bar{x}}$.
In a similar way, we show that

$$s - B^f_{\bar{x}} = B^f_{\bar{x}},$$

the proof is complete. □

Hypothesis: In what follows, we assume that E and F are metrizable and adopt the notation $A^f_{\bar{x}}$ for a lower level set.

Next, we prove the following elementary property.

PROPOSITION 4.3. *Let* $f: E \to F^\bullet$ *and* $\bar{x} \in \mathrm{Dom}\, f$. *Then,*

(1) $A^f_{\bar{x}} = A^f_{\bar{x}} - C$;
(2) $B^f_{\bar{x}} = B^f_{\bar{x}} + C$.

Proof. (1) The inclusion $A^f_{\bar{x}} \subseteq A^f_{\bar{x}} - C$ is clear. For the converse inclusion, let $y \in A^f_{\bar{x}}$ and $c \in C$. Let V be a neighborhood of $y - c$. Then $V + c$ is a neighborhood of y. Hence there is a neighborhood U of $\bar{x}$ such that $f(U) \subseteq V + c + C \cup \{+\infty\} \subseteq V + C \cup \{+\infty\}$. By this $y - c \in A^f_{\bar{x}}$.

(2) The second equality can be established by inverting the order. □

THEOREM 4.4. *Let* $f : E \to F^\bullet$ *and* $\bar{x} \in \operatorname{Dom} f$. *Then,*

(1) f *is l.s.c at* $\bar{x} \Longleftrightarrow A_{\bar{x}}^{f} = f(\bar{x}) - C$;
(2) f *is u.s.c at* $\bar{x} \Longleftrightarrow B_{\bar{x}}^{f} = f(\bar{x}) + C$.

Proof. ($\Rightarrow$) Suppose that f is l.s.c at $\bar{x}$ and $y \in A_{\bar{x}}^{f}$. We can easily see that (as detailed in Proposition 5.1),

$$y - f(\bar{x}) \in -C.$$

Therefore

$$A_{\bar{x}}^{f} \subset f(\bar{x}) - C. \tag{4.3}$$

Since f is l.s.c at $\bar{x}$, $f(\bar{x}) \in A_{\bar{x}}^{f}$, by Proposition 4.3, we have

$$f(\bar{x}) - C \subset A_{\bar{x}}^{f}. \tag{4.4}$$

We hence deduce, via (4.3) and (4.4), that

$$f(\bar{x}) - C = A_{\bar{x}}^{f}.$$

($\Leftarrow$) Assume that $A_{\bar{x}}^{f} = f(\bar{x}) - C$. As $0 \in C$, $f(\bar{x}) \in A_{\bar{x}}^{f}$. Following Proposition 4.1, f is lower semicontinuous at $\bar{x}$.

(2) The second equivalence can be established similarly. □

COROLLARY 4.5. *Let* $f : E \to F^\bullet$ *and* $\bar{x} \in \operatorname{Dom} f$. *Assume that* C *is pointed and* (F, C) *is normal. Then, the assertions below are equivalent.*

(i) f is continuous at $\bar{x}$;
(ii) $A_{\bar{x}}^{f} \cap B_{\bar{x}}^{f} \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii) is clear. Let us prove that (ii)$\Rightarrow$(i).
Let $y \in A_{\bar{x}}^{f} \cap B_{\bar{x}}^{f}$. Then, for each sequence $(x_n)_n$ that converges to $\bar{x}$, there exist two sequences $(a_n)_n$ and $(b_n)_n$ in F such that

$$\lim_{n \to +\infty} a_n = y \quad \text{and} \quad a_n \leqslant_c f(x_n), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n \to +\infty} b_n = y \quad \text{and} \quad f(x_n) \leqslant_c b_n, \quad \forall n \in \mathbb{N}.$$

In particular, for the stationary sequence $x_n = x$, for every $n \in \mathbb{N}$, there exists two sequences $(a'_n)_n$ and $(b'_n)_n$ in F such that

$$\lim_{n \to +\infty} a'_n = y \quad \text{and} \quad a'_n \leqslant_c f(\bar{x}), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n\to+\infty} b'_n = y \quad \text{and} \quad f(\bar{x}) \leqslant_c b'_n, \quad \forall n \in \mathbb{N}.$$

Since C is closed,

$$f(\bar{x}) - y \in C \cap (-C).$$

Since C is pointed, then $f(\bar{x}) = y \in A_{\bar{x}}^f \cap B_{\bar{x}}^f$. Following Proposition 4.1, f is l.s.c and u.s.c at $\bar{x}$. The normality of (F, C) ensures the continuity of f at $\bar{x}$. □

5. Properties of Lower and Upper Level Sets

This section is devoted to the study of the behavior of the lower and upper level sets. We show that they have remarkable properties, both topological and geometrical. We focus only on the lower level set, because the properties of the local upper level set can be deduced straightforwardly by similarity.

At first, we provide a bound for $A_{\bar{x}}^f$ and $B_{\bar{x}}^f$.

PROPOSITION 5.1. *Let $f: E \to F^\bullet$ and $\bar{x} \in \mathrm{Dom}\, f$. The following assertion holds:*

$$A_{\bar{x}}^f \leqslant_c \{f(\bar{x})\} \leqslant_c B_{\bar{x}}^f.$$

Proof. Let $y \in A_{\bar{x}}^f$ and consider the sequence $(x_n)_n$ given by $x_n = \bar{x}$ for every $n \in \mathbb{N}$. There exists a sequence $(b_n)_n$ in F such that

$$\lim_{n\to+\infty} b_n = y \qquad \text{and} \qquad b_n \leqslant_c f(\bar{x}), \quad \forall n \in \mathbb{N}.$$

Since C is closed, $y \leqslant_c f(\bar{x})$. Hence, $A_{\bar{x}}^f \leqslant_c \{f(\bar{x})\}$.
In the same manner, we show that $\{f(\bar{x})\} \leqslant_c B_{\bar{x}}^f$ by inverting the order. □

The useful property of directness is also verified. Precisely, we have the following:

PROPOSITION 5.2. *Let $f: E \to F^\bullet$ and $\bar{x} \in \mathrm{Dom}\, f$. Suppose that F is a Banach lattice. Then, $A_{\bar{x}}^f$ is directed upwards and $B_{\bar{x}}^f$ is directed downwards.*

Proof. Take $y_1, y_2 \in A_{\bar{x}}^f$ and $(x_n)_n$ a sequence in E converging to $\bar{x}$. Then there exist two sequences $(b_n)_n$ and $\left(b'_n\right)_n$ in F such that

$$\lim_{n\to+\infty} b_n = y_1 \qquad \text{and} \qquad b_n \leqslant_c f(x_n), \quad \forall n \in \mathbb{N} \tag{5.1}$$

and

$$\lim_{n\to+\infty} b_n^{'} = y_2 \quad \text{and} \quad b_n^{'} \leqslant_c f(x_n), \quad \forall n \in \mathbb{N}. \tag{5.2}$$

As the map :

$$\begin{aligned} \sup\colon F \times F &\to F \\ (x, y) &\mapsto \sup(x, y) \end{aligned}$$

is uniformly continuous (see [23] Proposition 5.2 p. 83), taking into consideration (5.1) and (5.2), we obtain

$$\lim_{n\to+\infty} \sup(b_n, b_n^{'}) = \sup(y_1, y_2) \quad \text{and} \quad \sup(b_n, b_n^{'}) \leqslant_c f(x_n), \quad \forall n \in \mathbb{N}.$$

It follows that $\sup(y_1, y_2) \in A_{\bar{x}}^{f}$. □

PROPOSITION 5.3. *For each* $\xi \in Y^*$, $\xi(\sup A_{\bar{x}}^{f}) = \sup \xi(A_{\bar{x}}^{f})$.

Proof. Let $\xi \in Y^*$. $A_{\bar{x}}^{f}$ being directed upwards, thanks to Proposition 4.1 of [2], it suffices to check that ξ is continuous with respect to the Scott topology. Since $\mathbb{R}$ is continuous (in the of sense of [2] see example 1.1 in [2] for further details) and ξ is lower semicontinuous continuous (because continuous), it follows from [Theorem 4.2, [2]] that ξ is Scott-continuous. The proof is then complete. □

PROPOSITION 5.4. *Let* $f : E \to F^{\bullet}$ *and* $\bar{x} \in \mathrm{Dom}\, f$. *Then* $A_{\bar{x}}^{f}$ *and* $B_{\bar{x}}^{f}$ *are convex.*

Proof. Take $y_1, y_2 \in A_{\bar{x}}^{f}$, $\lambda \in [0, 1]$ and $(x_n)_n$ a sequence in E that converges to $\bar{x}$. Then there exist two sequences $\left(b_n^{'}\right)_n$ and $\left(b_n^{''}\right)_n$ in F such that

$$\lim_{n\to+\infty} b_n^{'} = y_1 \quad \text{and} \quad b_n^{'} \leqslant_c f(x_n), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n\to+\infty} b_n^{''} = y_2 \quad \text{and} \quad b_n^{''} \leqslant_c f(x_n), \quad \forall n \in \mathbb{N}.$$

Take $b_n = \lambda b_n^{'} + (1-\lambda)\, b_n^{''}$. On one hand we have

$$\lim_{n\to+\infty} b_n = \lambda y_1 + (1-\lambda)\, y_2. \tag{5.3}$$

On the other hand,

$$\begin{aligned} f(x_n) - b_n &= f(x_n) - \lambda b_n^{'} - (1-\lambda)\, b_n^{''} \\ &= \lambda \left(f(x_n) - b_n^{'} \right) + (1-\lambda) \left(f(x_n) - b_n^{''} \right). \end{aligned}$$

As $f(x_n) - b_n^{'} \in C \cup \{+\infty\}$ and $f(x_n) - b_n^{''} \in C \cup \{+\infty\}$,

$$b_n \leqslant_c f(x_n), \ \ \forall n \in \mathbb{N} \tag{5.4}$$

we deduce from (5.3) and (5.4) that

$$\lambda y_1 + (1-\lambda)\, y_2 \in A_{\bar{x}}^{f}.$$

Similarly, we can prove that $B_{\bar{x}}^{f}$ is convex. □

6. The Objective of the Paper

The study of the l.s.c regularization of vector-valued mappings has been initiated by Théra for maps with values in complete (lattice) Daniell spaces. Our ambition here is to define a lower semiconductivity for of a vector-valued mapping $f: E \to F^{\bullet}$, when F is a complete Banach lattice.[3]

Assuming, throughout this section, that for every $\overline{x} \in \mathrm{Dom}\, f$, $A_{\bar{x}}^{f} \neq \emptyset$, let us agree to introduce the following mappings:

$$I_f(\overline{x}) := \sup A_{\bar{x}}^{f}.$$

Remark 6.1. F being a complete lattice, $A_{\bar{x}}^{f}$ is upper bounded. It follows that I_f is well defined i.e., sup $A_{\bar{x}}^{f}$ exists.

We first begin with the case where $F = H$ is a separable complete Hilbert lattice space. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of H. The order on H is defined by the closed convex cone given by

$$H_+ = \{x \in H \mid \langle e_p, x\rangle \geqslant 0, \ \forall p \in \mathbb{N}\}.$$

Clearly, the polar cone of H_+ is equal to H_+, i.e. , $H_+^* = H_+$.

In order to conclude the semicontinuity of I_f in this first case with the use of semicontinuity of the usual scalar lower limit, we need two ingredients.

LEMMA 6.2. *Let $f: E \to H^{\bullet}$ and $\bar{x} \in \mathrm{Dom}\, f$. Assume that $A_{\bar{x}}^{f} \neq \emptyset$. Then, for each $p \in \mathbb{N}$, one has*

$$\left\langle e_p, A_{\bar{x}}^{f} \right\rangle = A_{\bar{x}}^{\langle e_p, f\rangle},$$

where $\left\langle e_p, A_{\bar{x}}^{f} \right\rangle := \left\{ \langle e_p, y\rangle \mid y \in A_{\bar{x}}^{f} \right\}$.

[3]We can present the l.s.c regularization with quite different arguments in the case where F is a reflexive Banach space ordered by a normal cone. Here we restrict ourselves to the setting of complete Banach lattices.

Proof. *Let* $p\in\mathbb{N}, y\in A_{\bar{x}}^{f}$ *and* $(x_n)_n$ *be a sequence in* E *converging to* $\bar{x}$, *then there is a sequence* $(b_n)_n$ *in* H *such that*

$$\lim_{n\to+\infty} b_n = y \quad \text{and} \quad b_n \leqslant f(x_n), \quad \forall n\in\mathbb{N},$$

which implies that

$$\lim_{n\to+\infty} \langle e_p, b_n\rangle = \langle e_p, y\rangle \quad \text{and} \quad \langle e_p, b_n\rangle \leqslant \langle e_p, f(x_n)\rangle, \quad \forall n\in\mathbb{N}$$

and therefore $\langle e_p, y\rangle \in A_{\bar{x}}^{\langle e_p, f\rangle}$. *It follows that*

$$\left\langle e_p, A_{\bar{x}}^{f}\right\rangle \subset A_{\bar{x}}^{\langle e_p, f\rangle} \quad \text{for each} \quad p\in\mathbb{N}. \tag{6.1}$$

Let us show the converse inclusion: $A_{\bar{x}}^{\langle e_p, f\rangle} \subset \langle e_p, A_{\bar{x}}^{f}\rangle$ *for each* $p\in\mathbb{N}$. *So, let* $(x_n)_n$ *be a sequence in* E *converging to* $\bar{x}, p\in\mathbb{N}$ *and* $y\in A_{\bar{x}}^{\langle e_p, f\rangle}$. *Consider* $(y_n)_n$ *in* $\mathbb{R}$ *such that*

$$\lim_{n\to+\infty} y_n = y \quad \text{and} \quad y_n \leqslant \langle e_p, f(x_n)\rangle, \ \forall n\in\mathbb{N}. \tag{6.2}$$

Let $z_0\in A_{\bar{x}}^{f}$ *be fixed and take* $z := z_0 + \left(y - z_0^{p}\right) e_p$, *where* $z_0^{p} = \langle z_0, e_p\rangle$. *Let us prove that* $z\in A_{\bar{x}}^{f}$. *As* $z_0\in A_{\bar{x}}^{f}$, *there exists* $(z_n)_n$ *such that*

$$\lim_{n\to+\infty} z_n = z_0 \ \text{and} \quad z_n \leqslant_{H_+} f(x_n), \quad \forall n\in\mathbb{N}. \tag{6.3}$$

Set $z_n' = z_n + \left(y_n - z_n^{p}\right) e_p$ *with* $z_n^{p} = \langle z_n, e_p\rangle$ *and observe that* $\lim_{n\to+\infty} z_n' = z$. *We shall check that*

$$z_n' \leqslant_{H_+} f(x_n), \quad \forall n\in\mathbb{N}. \tag{6.4}$$

Let $q\in\mathbb{N}$. *Remarking that* $\langle e_q, f(x_n) - z_n'\rangle = \langle e_q, f(x_n) - z_n\rangle$, *it follows from (6.3) that* $\langle e_q, f(x_n) - z_n'\rangle \geqslant 0$. *Now for* $q = p$, *a simple computation shows that* $\langle e_q, f(x_n) - z_n'\rangle = \langle e_p, f(x_n) - y_n\rangle$, *then by (6.2) we have* $\langle e_q, f(x_n) - z_n'\rangle \geqslant 0$. *Hence, (6.4) is satisfied and therefore* $z\in A_{\bar{x}}^{f}$. *As* $\langle e_p, z\rangle = y$, *we deduce that*

$$A_{\bar{x}}^{\langle e_p, f\rangle} \subset \langle e_p, A_{\bar{x}}^{f}\rangle. \tag{6.5}$$

Using (6.1) *and* (6.5) *we obtain* $A_{\bar{x}}^{\langle e_p, f\rangle} = \langle e_p, A_{\bar{x}}^{f}\rangle$ *for each* $p\in\mathbb{N}$. □

LEMMA 6.3. [19] *Let* $Y = \prod_i Y_i$ *be the space product of a family of ordered vector spaces* Y_i. *Then, a map* $f: E\to Y$ *is l.s.c, if and only if, its projections* $f_i = p_i \circ f$ *are l.s.c.*

Here, p_i denotes the projection from Y into Y_i.

THEOREM 6.4. *Let $f: E \to H^\bullet$ and let $\overline{x} \in \operatorname{Dom} f$. Then, I_f is l.s.c at $\overline{x}$.*

Proof. Since $A_{\bar{x}}^f$ is directed, one can easily check that for all $p \geqslant 0$, we have

$$\langle e_p, I_f(\overline{x})\rangle = \langle e_p, \sup A_{\bar{x}}^f\rangle = \sup\langle e_p, A_{\bar{x}}^f\rangle.$$

Then, using Lemma 6.2 and Remark 3.3, we deduce that

$$\langle e_p, I_f(\overline{x})\rangle = \sup A_{\bar{x}}^{f_p} = \liminf_{x\to\overline{x}} f_p(x),$$

where $f_p = \langle e_p, f\rangle$. Clearly, $\langle e_p, I_f\rangle$ is l.s.c for each p, thus f is l.s.c. □

Next, we present the more general case where F is a complete Banach lattice in which we do not make recourse to semicontinuity of the scalar lower limit.

THEOREM 6.5. *Suppose F is a complete Banach lattice and $A_{\bar{x}}^f \neq \emptyset$ for all $\overline{x} \in \operatorname{Dom} f$. Then, I_f is lower semicontinuous at every $x \in \operatorname{Dom} f$, and therefore I_f defines a l.s.c regularization of f.*

We prove first some technical Lemmata that will be useful for proving the main result.

LEMMA 6.6. *For every convex cone C in F, we have*

$$(F \setminus C) - C = F \setminus C. \tag{6.6}$$

Proof. The proof is standard and based on the convexity of C. □

LEMMA 6.7. *We have $I_f(\bar{x}) \in cl\, A_{\bar{x}}^f$.*

Proof. It suffices to show that for each $d \in \operatorname{Int} C$, one has $I_f(\bar{x}) - d \in A_{\bar{x}}^f$. Indeed, suppose that $I_f(\bar{x}) - d \notin A_{\bar{x}}^f$ for some $d \in \operatorname{Int} C$. Then we may separate the point $I_f(\bar{x}) - d$ and the convex set $A_{\bar{x}}^f = A_{\bar{x}}^f - C$, i.e., there is some $y^* \in F^*\{0\}$ such that

$$\langle y^*, I_f(\bar{x}) - d\rangle \geqslant \langle y^*, z\rangle,\ \forall z \in A_{\bar{x}}^f - C.$$

It easily follows that $y^* \in C^*$ and

$$\langle y^*, I_f(\bar{x})\rangle \geqslant \langle y^*, d\rangle + \langle y^*, z\rangle, \quad \forall z \in A_{\bar{x}}^f.$$

Since $\langle y^*, d\rangle > 0$, the above inequality leads to a contradiction with Proposition 5.3. □

As a consequence of the previous Lemma we derive the following technical Lemma.

LEMMA 6.8. *For each* $\overline{y} \in A^f_{\overline{x}}$ *such that* $\overline{y} <_c I_f(\bar{x})$, *there exists a sequence* $(\beta_k)_k$ *in cl* $A^f_{\overline{x}}$ *such that*

$$\beta_k \to I_f(\overline{x}) \text{ as } n \text{ goes to } +\infty \text{ and } \overline{y} <_c \beta_k, \text{ for all } k. \tag{6.7}$$

Now, for $\overline{x} \in \operatorname{Dom} f$, we introduce the following sets

$$E_f(\overline{x}) := \{y \in A^f_{\overline{x}} \mid y <_c I_f(\overline{x})\} \tag{6.8}$$

and

$$H_f(\overline{x}) := \{y \in \operatorname{cl} A^f_{\overline{x}} \mid y \leqslant_c I_f(\overline{x})\} \tag{6.9}$$

and establish the following:

LEMMA 6.9. *We have cl* $E_f(\overline{x}) = H_f(\overline{x})$.

Proof. Let $y \in \operatorname{cl}(E_f(\overline{x}))$. There exists a sequence $(y_k)_k$ converging to y such that $y_k \in E_f(\overline{x})$, for all k. Clearly, $y \in \operatorname{cl} A^f_{\overline{x}}$ and

$$I_f(\overline{x}) - y_k \in \operatorname{Int} C \subset C \ \ \forall k. \tag{6.10}$$

Passing to the limit in (6.10) whenever k goes to $+\infty$, we obtain that

$$I_f(\overline{x}) - y \in C,$$

or equivalently $y \in H_f(\overline{x})$.

Conversely, let $\overline{y} \in H_f(\overline{x})$. Then, $\overline{y} \in \operatorname{cl} A^f_{\overline{x}}$ and $I_f(\overline{x}) - \overline{y} \in C$. Therefore, there exists $(y_k)_k$ in $A^f_{\overline{x}}$ such that $y_k \to \overline{y}$. Consider now $(\nu_k)_k$ in $\operatorname{Int} C$ such that $\nu_k \to 0$. Take $\overline{y}_k := y_k - \nu_k$ and observe that $(\overline{y}_k)_k$ also converges to $\overline{y}$. On the other hand, Proposition 4.3 implies that

$$\overline{y}_k \in A^f_{\overline{x}} - \operatorname{Int} C \subset A^f_{\overline{x}} - C = A^f_{\overline{x}}. \tag{6.11}$$

Remark also that

$$\begin{aligned} I_f(\overline{x}) - \overline{y}_k &= (I_f(\overline{x}) - y_k) + \nu_k \\ &\in C + \operatorname{Int} C = \operatorname{Int} C. \end{aligned}$$

Thus, $\overline{y} \in \operatorname{cl} E_f(\overline{x})$. The proof of the Lemma is therefore established. □

Remark 6.10. Remark that $A^f_{\bar{x}} \subset \mathrm{cl}\, E_f(\overline{x}) = H_f(\overline{x}) = \mathrm{cl}\, A^f_{\bar{x}}$.

The following result will play a key role to derive the lower semicontinuity of the lower limit.

LEMMA 6.11. *For every sequence $(x_k)_k$ converging to $\overline{x}$, the sequence $(A^f_{x_k})_k$ is lower convergent to $A^f_{\bar{x}}$ in the sense of Painlevé–Kuratowski.*

Proof. We have to prove that each neighborhood of any point of the set $A^f_{\bar{x}}$ meets the sets $A^f_{x_k}$ for k sufficiently large. Suppose to the contrary that there are some point $y \in A^f_{\bar{x}}$ and a neighborhood V of the origin such that $(y+V) \cap A^f_{x_k} = \emptyset$ for all $k \geqslant 1$. In particular, there is some $d \in \mathrm{Int}\, C$ such that $(y-d+C) \cap A^f_{\bar{x}} = \emptyset$ for all $k \geqslant 1$. Since $y \in A^f_{\bar{x}}$, there is $(y_k)_k$ converging to y such that $y_k \leqslant_c f(x_k)$. We may assume that $y_k \in y - \frac{1}{2}d + C$ for $k \geqslant 1$, which implies also

$$f(x_k) \in y - \frac{1}{2}d + C, \quad \forall k \geqslant 1.$$

Moreover, as $y - d \notin A^f_{x_k}$, for every k, there is a sequence $(x_{k,n})_n$ converging to x_k such that $y - \frac{n-1}{n}d \not\leqslant_c f(x_{k,n})$. Choose a subsequence $(x_{k,n_k})_k$ converging to $\bar{x}$. Since $y \in A^f_{\bar{x}}$, there is a sequence $(y_{k,n_k})_k$ converging to y such that $y_{k,n_k} \leqslant_c f(x_{k,n_k})$. One may assume that $y_{n,n_k} \in y - \frac{1}{2}d + C$ for all $k > 1$. Then for all these k, $y - \frac{1}{2}d \leqslant_c f(x_{k,n_k})$, which is in contradiction with $y - \frac{n_k-1}{n_k}d \not\leqslant_c f(x_{k,n_k})$ when k is sufficiently large. □

We are now ready to establish the main result of the paper.

Proof. of theorem 6.5. Let $\overline{x} \in \mathrm{Dom}\, f$ and let $(x_n)_n$ be a sequence converging to $\overline{x}$. Thanks to Lemma 6.7, there exists a sequence $(y_k)_k$ converging to $I_f(\overline{x})$ such that $y_k \in A^f_{\bar{x}}$. From Lemma 6.11, it follows that $y_k \in \liminf_n A^f_{x_n}$. Select $y^n_k \in A^f_{x_n}$ such that $(y^n_k)_n$ converges to y_k as n goes to $+\infty$. Following the beginning of the proof of Lemma 6.11 we can assume that $(y^n_k)_n$ converges uniformly in k to y_k. Indeed, we can consider $y^n_k = y_k - v_n$ with $v_n \in \mathrm{Int}\, C$ such that

$$\lim_{n \to +\infty} v_n = 0.$$

Clearly,

$$y^n_k \leqslant_c \sup A^f_{x_n} = I_f(x_n)$$

This being true for all k, let $k(n)$ be a map such that $k(n) \to +\infty$. Take $b_n = y^n_{k(n)}$; the $(b_n)_n$ sequence converges to $I_f(\overline{x})$ (thanks to the uniform convergence of $(y^n_k)_n$ to y_k) and satisfies

$$b_n \leqslant_c I_f(x_n).$$

The proof is complete. □

Remark 6.12. From now on, we will use the following notation

$$v - \liminf_{x \to \overline{x}} f(x) := I_f(\overline{x}) = \sup A^f_{\overline{x}}$$

COROLLARY 6.13. *The assertions below are equivalent*

- f *is lower semicontinuous at* $\overline{x} \in \mathrm{Dom}\, f$;
- $f(\overline{x}) \leqslant_c v - \liminf_{x \to \overline{x}} f(x)$.

Proof. By Proposition 4.1, f is l.s.c at $\overline{x}$ if, and only if $f(\overline{x}) \in A^f_{\overline{x}}$. Then, if f is l.s.c at $\overline{x}$, as $v - \liminf_{x \to \overline{x}} f(x)$ is the least upper bound of $A^f_{\overline{x}}$, it follows that

$$f(\overline{x}) \leqslant_c v - \liminf_{x \to \overline{x}} f(x). \tag{6.12}$$

Conversely, suppose that (6.12) holds. We know that $f(\overline{x})$ is an upper bound of $A^f_{\overline{x}}$, a fortiori

$$v - \liminf_{x \to \overline{x}} f(x) \leqslant_c f(\overline{x}).$$

Hence, as the cone C is pointed, we derive

$$v - \liminf_{x \to \overline{x}} f(x) = f(\overline{x}).$$

Therefore, thanks to the characterization of semicontinuity of [6] and our main result, we deduce that f is necessarily l.s.c at $\overline{x}$. Indeed, having in mind

$$I_f(z) := v - \liminf_{x \to z} f(x) \leqslant_c f(z), \quad \forall z \in E,$$

for any sequence $(x_n)_n$ such that $x_n \to \overline{x}$, by Theorem 6.5, there exists a sequence $(b_n)_n$ converging to $I_f(\overline{x}) = f(\overline{x})$ such that

$$b_n \leqslant_c I_f(x_n) \leqslant_c f(x_n) \quad \forall n.$$

The proof is established. □

Remark 6.14. Similarly, we can define the upper semicontinuous regularization of f by

$$v-\limsup_{x\to\bar{x}} f(x) := \inf B_{\bar{x}}^{f}.$$

7. Compatibility with the Standard Cases

Now, we derive from the main result a more flexible proof for the well known result that says: for every real-valued function, the lower limit is lower semicontinuous. We consider here, more generally, finite dimensional-valued mappings.

Consider $F=\mathbb{R}^p$, $f=(f_1, f_2, \ldots, f_p)$ and $\bar{x}\in \operatorname{Dom} f$. Note that the order in this case goes back to the usual order of $\mathbb{R}$, generated by the cone $C:=\mathbb{R}_+^p$ and will be simply denoted by $\leqslant$.
We claim that

$$\operatorname{cl} A_{\bar{x}}^{f} = \prod_{i=1}^{p}]-\infty, \liminf_{x\to\bar{x}} f_i(x)],$$

and

$$\operatorname{cl} B_{\bar{x}}^{f} = \prod_{i=1}^{p} [\limsup_{x\to\bar{x}} f_i(x), +\infty[.$$

In fact, consider first the case $p=1$. Let $f:E\to\mathbb{R}$, $\bar{x}\in \operatorname{Dom} f$, $y\in A_{\bar{x}}^{f}$ and $(x_n)_n$ be a sequence converging to $\bar{x}$. There exists $(b_n)_n$ such that

$$\lim_{n\to+\infty} b_n = y \qquad \text{and} \qquad b_n \leqslant f(x_n), \ \forall n\in\mathbb{N}.$$

This yields $y\leqslant \liminf_{x\to\bar{x}} f(x)$ and thus

$$y\in \left]-\infty, \liminf_{x\to\bar{x}} f(x)\right],$$

whence, $\operatorname{cl} A_{\bar{x}}^{f} \subset]-\infty, \liminf_{x\to\bar{x}} f(x)]$.

Thanks to Proposition 4.3, $A_{\bar{x}}^{f}$ is an interval of $\mathbb{R}$ containing $-\infty$, then to prove the second inclusion, it suffices to show that

$$\liminf_{n\to+\infty} f(x_n) \in \operatorname{cl} A_{\bar{x}}^{f}.$$

To this end, let $\varepsilon > 0$ and $(x_n)_n$ be a sequence in E converging to $\bar{x}$; there exists $N \in \mathbb{N}$ such that

$$\liminf_{n\to+\infty} f(x_n) - \varepsilon \leqslant f(x_n) \qquad \text{for each } n \geqslant N.$$

Set

$$b_n := \begin{cases} -\infty & \text{if } \ n \leqslant N \\ \liminf_{n\to+\infty} f(x_n) - \varepsilon & \text{if } n > N. \end{cases}$$

We have $\lim_{n\to+\infty} b_n = \liminf_{n\to+\infty} f(x_n) - \varepsilon$ and $b_n \leqslant f(x_n)$ for every $n \in \mathbb{N}$. This leads to

$$\liminf_{n\to+\infty} f(x_n) - \varepsilon \in A_{\bar{x}}^{f}$$

for all $\varepsilon > 0$, accordingly

$$]-\infty, \liminf_{x\to\bar{x}} f(x)] \subset \operatorname{cl} A_{\bar{x}}^{f}.$$

Thus,

$$]-\infty, \liminf_{x\to\bar{x}} f(x)] = \operatorname{cl} A_{\bar{x}}^{f}.$$

Let us show this inequality for $p > 1$. Let f be a mapping defined from E into $\mathbb{R}^p$ where $f = (f_1, f_2, \ldots, f_p)$. We observe that

$$\begin{aligned} A_{\bar{x}}^{f} &= \{ y \in \mathbb{R}^p \mid \forall (x_n) \to \bar{x},\ \exists (b_n) \to y \mid b_n \leqslant f(x_n)\ \ \forall n \in \mathbb{N} \} \\ &= \left\{ (y_1, \ldots, y_p) \in \mathbb{R}^p \mid \forall (x_n) \to \bar{x},\ \exists (b_n^i) \underset{n\to+\infty}{\to} y_i \mid b_n^i \leqslant f_i(x_n)\ \ \forall n \in \mathbb{N} \right\}, \end{aligned}$$

for $i = 1, 2, \ldots, p$. This yields

$$\begin{aligned} A_{\bar{x}}^{f} &= \left\{ (y_1, \ldots, y_p) \in \mathbb{R}^p \mid y_i \in A_{\bar{x}}^{f_i} \ \text{for } i = 1, 2, \ldots, p \right\} \\ &= \prod_{i=1}^{p} A_{\bar{x}}^{f_i}. \end{aligned}$$

This product being finite, it follows that

$$\begin{aligned} \operatorname{cl} A_{\bar{x}}^{f} &= \operatorname{cl} \prod_{i=1}^{p} A_{\bar{x}}^{f_i} \\ &= \prod_{i=1}^{p}]-\infty, \liminf_{x\to\bar{x}} f_i(x)]. \end{aligned}$$

establishing the proof. □

Now, we recapture the classical result for finite dimensional valued-functions:

THEOREM 7.1. *The lower limit for finite dimensional valued-functions is lower semicontinuous.*

Proof. Take $F=\mathbb{R}^p$. It suffices to take $C=\mathbb{R}^p_+$ and replace $\leqslant_c$ by the usual order $\leqslant$ in the proof of the main result. □

Remark 7.2. Notice here that our proof of this classical result is independent of the closure of the epigraph.

8. Extension of Usual Operations

To extend the usual operations on the lower and upper limits to the vector case, we first establish the relation between the lower level set of the sum of two maps and the sum of their lower level sets.

PROPOSITION 8.1. *Let f and h be two vector-valued mappings from E into $F^\bullet$ and $\bar{x}\in \operatorname{Dom} f\cap \operatorname{Dom} h$. Then the following inclusions hold:*

(1) $A^f_{\bar{x}}+A^h_{\bar{x}}\subset A^{f+h}_{\bar{x}}$;
(2) $B^f_{\bar{x}}+B^h_{\bar{x}}\subset B^{f+h}_{\bar{x}}$.
(3) *If moreover, f or h is continuous at $\bar{x}$, the inclusions in* (1) *and* (2) *become equalities.*

Proof. (1) Let $y\in A^f_{\bar{x}}+A^h_{\bar{x}}$, there exist $y_1\in A^f_{\bar{x}}$ and $y_2\in A^h_{\bar{x}}$ such that $y=y_1+y_2$. Let $(x_n)_n$ be a sequence converging to $\bar{x}$. As $y_1\in A^f_{\bar{x}}$ and $y_2\in A^h_{\bar{x}}$, there exist $\left(b^{'}_n\right)_n$ and $\left(b^{''}_n\right)_n$ in F such that

$$\lim_{n\to+\infty} b^{'}_n=y_1 \quad \text{and} \quad b^{'}_n\leqslant f(x_n), \quad \forall n\in\mathbb{N}$$

and

$$\lim_{n\to+\infty} b^{''}_n=y_2 \quad \text{and} \quad b^{''}_n\leqslant h(x_n), \quad \forall n\in\mathbb{N}\ .$$

Set $b_n=b^{'}_n+b^{''}_n$, we have $\lim_{n\to+\infty} b_n=y_1+y_2=y$, and

$$b_n=b^{'}_n+b^{''}_n\leqslant f(x_n)+h(x_n), \quad \forall n\in\mathbb{N}.$$

It follows that $y\in A^{f+h}_{\bar{x}}$.

(2) The second inclusion can be established as in (1).
(3) Suppose now that f is continuous at $\bar{x}$ and consider $(x_n)_n$ converging to $\bar{x}$ and $y \in A_{\bar{x}}^{f+h}$. There exists a sequence $(b_n)_n$ in F such that

$$\lim_{n \to +\infty} b_n = y \quad \text{and} \quad b_n \leqslant f(x_n) + h(x_n), \quad \forall n \in \mathbb{N}.$$

This yields

$$b_n - f(x_n) \leqslant h(x_n), \quad \forall n \in \mathbb{N}. \tag{8.1}$$

Let us take

$$a_n = \begin{cases} b_n - f(x_n) & \text{if } x_n \in \operatorname{Dom} f \\ h(x_n) & \text{otherwise.} \end{cases}$$

f being continuous at $\bar{x}$, it follows for n large enough, that $x_n \in \operatorname{Dom} f$, and therefore

$$\lim_{n \to +\infty} a_n = y - f(\bar{x}). \tag{8.2}$$

Hence, we deduce from (8.1) and (8.2) that $y - f(\bar{x}) \in A_{\bar{x}}^h$. Thus, $y \in f(\bar{x}) + A_{\bar{x}}^h$.

As f is continuous at $\bar{x}$, f is s-s.c.i at $\bar{x}$. It follows from Proposition 4.1 that $f(\bar{x}) \in A_{\bar{x}}^f$, then

$$y \in A_{\bar{x}}^f + A_{\bar{x}}^h.$$

The second inclusion follows analogously. □

THEOREM 8.2. *Let f and h be two vector-valued mappings from E into $F^\bullet$ and $\bar{x} \in \operatorname{Dom} f \cap \operatorname{Dom} h$. The following assertions hold.*

(1) $v - \liminf_{y \to \bar{x}} f(y) + v - \liminf_{y \to \bar{x}} h(y) \leqslant_c v - \liminf_{y \to \bar{x}} (f+h)(y)$;
(2) $v - \limsup_{y \to \bar{x}} (f+h)(y) \leqslant_c v - \limsup_{y \to \bar{x}} f(y) + v - \limsup_{y \to \bar{x}} h(y)$.
(3) *If moreover f or h is continuous at $\bar{x}$, the inequalities in (1) and (2) become equalities.*

Proof. (1) By virtue of Proposition 8.1, we have $A_{\bar{x}}^f + A_{\bar{x}}^h \subset A_{\bar{x}}^{f+h}$. Let $z \in A_{\bar{x}}^f$ fixed; then $z + A_{\bar{x}}^h \subset A_{\bar{x}}^{f+h}$, and therefore

$$\sup\left(z + A_{\bar{x}}^h\right) \leqslant \sup A_{\bar{x}}^{f+h}.$$

Hence,

$$z + v - \liminf_{y \to \bar{x}} h(y) \leqslant v - \liminf_{y \to \bar{x}} (f + h)(y),$$

for all $z \in A_{\bar{x}}^{f}$. Then

$$\sup_{z \in A_{\bar{x}}^{f}} \left(z + v - \liminf_{y \to \bar{x}} h(y) \right) \leqslant v - \liminf_{y \to \bar{x}} (f + h)(y),$$

which leads to

$$\sup_{z \in A_{\bar{x}}^{f}} z + v - \liminf_{y \to \bar{x}} h(y) \leqslant v - \liminf_{y \to \bar{x}} (f + h)(y).$$

Accordingly,

$$v - \liminf_{y \to \bar{x}} f(y) + v - \liminf_{y \to \bar{x}} h(y) \leqslant v - \liminf_{y \to \bar{x}} (f + h)(y).$$

(2) The second inequality can be done similarly.
(3) Assume that f is continuous at $\bar{x}$, using Proposition 8.1, we obtain

$$A_{\bar{x}}^{f} + A_{\bar{x}}^{h} = A_{\bar{x}}^{f+h}.$$

Thanks to Theorem 4.4, we deduce that

$$f(\bar{x}) - C + A_{\bar{x}}^{h} = A_{\bar{x}}^{f+h}.$$

Proposition 4.3 implies that

$$A_{\bar{x}}^{h} - C = A_{\bar{x}}^{h},$$

which allows to say that

$$f(\bar{x}) + A_{\bar{x}}^{h} = A_{\bar{x}}^{f+h},$$

whence

$$\begin{aligned} \sup\left(A_{\bar{x}}^{f+h}\right) &= \sup\left(f(\bar{x}) + A_{\bar{x}}^{h}\right) \\ &= f(\bar{x}) + \sup\left(A_{\bar{x}}^{h}\right). \end{aligned}$$

Then,

$$f(x) + v - \liminf_{y \to \bar{x}} h(y) = v - \liminf_{y \to \bar{x}} (f + h)(y).$$

Since f is continuous, Theorem 4.11 implies that $A^f_{\bar{x}} = f(\bar{x}) - C$, and therefore $\sup A^f_{\bar{x}} = f(x)$. We conclude that

$$v - \liminf_{y \to x} f(y) + v - \liminf_{y \to x} h(y) = v - \liminf_{y \to x} (f + h)(y). \qquad \square$$

9. Application to Vector-Valued D.C. Mappings

In this section, H is as in Section 6. We shall apply our main result to show that every vector-valued D.C. mapping finite and continuous defined on a Banach space with values in H admits a continuous D.C. decomposition.

We first prove that each finite and continuous vector-valued D.C. mapping admits a lower semicontinuous D.C. decomposition.

Let Ω be a convex open of E. Recalling, for a mapping from Ω into H, the notation $I_\varphi(x) := \sup\left(A^\varphi_x\right)$, we state the following:

PROPOSITION 9.1. *Let $f : \Omega \to H$ be finite and continuous D.C. vector-valued mapping on Ω. If (g, h) is a D.C. decomposition of f on Ω, then $\left(I_g, I_h\right)$ is a D.C. decomposition of f on Ω.*

Proof. Let g be an H_+-convex. At first, we claim that I_g is H_+-convex. In fact, for $\bar{x} \in \operatorname{Dom} g$, as $A^g_{\bar{x}}$ is directed upwards, it follows that

$$\left\langle e_p, I_g(\bar{x}) \right\rangle = \left\langle e_p, \sup A^g_{\bar{x}} \right\rangle = \sup \left\langle e_p, A^g_{\bar{x}} \right\rangle, \quad \forall p \in \mathbb{N}.$$

According to Lemma 6.2, we have

$$\sup \left\langle e_p, A^g_{\bar{x}} \right\rangle = \sup A^{\langle e_p, g \rangle}_{\bar{x}} = \liminf_{x \to \bar{x}} \left\langle e_p, g(x) \right\rangle, \quad \forall p \in \mathbb{N}.$$

Therefore,

$$\text{epi}\left(\left\langle e_p, Ig \right\rangle\right) = \overline{\text{epi}\left\langle e_p, g \right\rangle}, \quad \forall p \in \mathbb{N}.$$

Since g is H_+-convex, for each $p \in \mathbb{N}$, $\left\langle e_p, g \right\rangle$ is convex. This yields $\left\langle e_p, I_g \right\rangle$ is convex for each $p \in \mathbb{N}$, i.e., I_g is H_+-convex.
Now, let (g, h) be a D.C. decomposition of f. Let us show that $\left(I_g, I_h\right)$ is a D.C. decomposition of f on Ω. For this, let $\bar{x} \in \Omega$, we have

$$A^{f+h}_{\bar{x}} = A^g_{\bar{x}}.$$

As f is continuous at $\bar{x}$, by Proposition 8.1, we obtain

$$A^f_{\bar{x}} + A^h_{\bar{x}} = A^g_{\bar{x}}.$$

On the other hand, Theorem 4.4 leads to

$$f(\bar{x}) - C + A_{\bar{x}}^{h} = A_{\bar{x}}^{g},$$

and from Proposition 4.3 we have

$$f(\bar{x}) + A_{\bar{x}}^{h} = A_{\bar{x}}^{g}.$$

Therefore

$$\begin{aligned}\sup\left(A_{\bar{x}}^{g}\right) &= \sup\left(f(\bar{x}) + A_{\bar{x}}^{h}\right)\\ &= f(\bar{x}) + \sup\left(A_{\bar{x}}^{h}\right).\end{aligned}$$

Hence, $f(\bar{x}) + I_h(\bar{x}) = I_g(\bar{x})$ for every $\bar{x} \in \Omega$. The proof is complete. □

In the scalar case it has been shown [8] that every real D.C. continuous function admits a continuous D.C. decomposition. Here, we provide a generalization for continuous vector-valued maps from a Banach space into a separable Hilbert space.

THEOREM 9.2. *Every finite, continuous and H_+-D.C. mapping on a convex and open subset Ω of E into a separable Hilbert space ordered by H_+ admits a continuous H_+-D.C. decomposition on Ω.*

Proof. Let (g, h) be an H_+-D.C. decomposition of f. Fix a point $x \in \Omega$. According to Proposition 9.1, (I_g, I_h) is a lower semicontinous H_+-D.C. decomposition on f on Ω. On the other hand, f is continuous at x. Then,

$$x \in \mathrm{Int}\,(\mathrm{Dom}\, f) = \mathrm{Int}\,(\mathrm{Dom}\, I_g \cap \mathrm{Dom}\, I_h) \subset [\mathrm{Dom}\, I_g \cap \mathrm{Int}\,(\mathrm{Dom}\, I_h)].$$

The map I_h being H_+-convex, lower semicontinuous and proper on Ω and

$$x \in \Omega \cap \mathrm{Int}\,(\mathrm{Dom}\, h),$$

it follows, for each p, that $I_{h_p} : H \to \mathbb{R}$ is convex, lower semicontinuous and proper. Hence, as $\mathbb{R}$ is normal, by Theorem 10.1 below we can conclude that I_{h_p} is continuous at x, for each p and then so is I_h. As $g(y) = f(y) + h(y)$ for each $y \in \Omega$, it results that I_g is continuous at x. The proof is complete. □

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Appendix A

THEOREM 10.1. *[24] Let $f : E \to F^{\bullet}$ be a vector mapping. Suppose that F is normal and f is C-convex, s-l.s.c and proper. If* $\mathrm{Int}\,(\mathrm{Dom}\, f)$ *is nonempty, then f is continuous on* $\mathrm{Int}\,(\mathrm{Dom}\, f)$.

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